Neurons in mean-field interaction

ICMNS 2015, June 8-10

F. Delarue, J. Inglis, S. Rubenthaler, E. Tanré

Université Nice Sophia Antipolis, INRIA Sophia, Université Nice Sophia Antipolis, INRIA Sophia

"Integrate and fire" model

The process X_t is the potential of one neuron. We want a model with the following features:

- when X_t reaches 1, it jumps back to 0 (the neuron "fires"),
- the neuron interacts with other neurons having the same law.

We set $M_t = \#\{s \leq t\,,\, X_{s-} = 1\}$. With N neurons:

$$X_t^i = \int_0^t b(X_s^i)ds + W_t^i - M_t^i + \frac{\alpha}{N}\sum_{j \neq i} M_t^j\,,\ \forall i\,. \qquad (1)$$

Limit equation when $N \to +\infty$,

$$X_t = \int_0^t b(X_s)ds + W_t - M_t + \alpha\mathbb{E}(M_t)\,. \qquad (2)$$

Questions

- Limit when $N \to +\infty$ (of (1)).
- Existence of a solution to (2), at any time?
- Synchronisation of neurons (could they fire at the same time?).

Results

If $e : t \mapsto \mathbb{E}(M_t)$ is differentiable, we can rewrite the equation into

$$dX_t = b(X_t)dt + \alpha e'(t)dt + dW_t \,.$$

Theorem

(Carceres, Carillo, Perthame) For all $\alpha > 0$, there exists $x_0 < 1$ (and t_0) such that $e'(t) \to +\infty$ when $t \to t_0-$.

Theorem

([DIRT15b]) For X_0 fixed, there exists (explicit) α_0 such that

$$\alpha < \alpha_0 \Rightarrow \text{solution up to any time.}$$

- existence and unicity in short time
- a priori estimates of $\|e\|_\infty$
- iteration of the short time proof

Short time

We show that

$\mathbb{P}(X_0 \in dy) \leq \beta(1-y)dy$ for y near 1 $\Rightarrow \exists!$ solution $[0; T(\alpha,\beta)]$.

The solution is the fixed point of

$$e \in \mathcal{C}^1[0;1] \mapsto \Gamma(e)(t) = \mathbb{E}(\sum_{s \leq t} \mathbb{1}_{X_{s-}=1})$$

where X is solution of

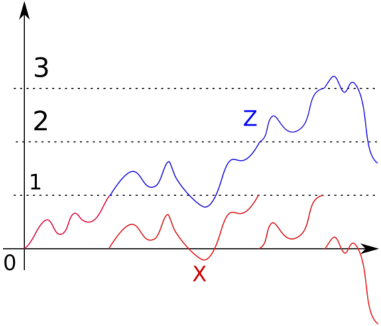
$$X_t = \int_0^t b(X_s)ds + W_t + \alpha e(t) - M_t \,.$$

Continuity

Set $e(t) = \mathbb{E}(M_t)$. We can write

$$Z_t = X_t + M_t \, , \; M_t = \lfloor (Z_t)^*_+ \rfloor$$

We have $X_0 < 1 - \epsilon \Rightarrow p(t, y) \leq C(\epsilon, \alpha)$



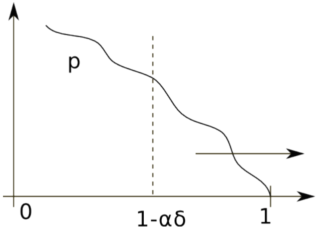
$$Z_t = X_0 + \int_0^t b(Z_s - M_s)ds + W_s + \alpha \mathbb{E}(M_t)$$

Continuity

Suppose $\Delta e(t) := e(t) - e(t-) = \delta > 0$, as $p(t, y) \leq C$ then the mass pushed over the treshold is

$$\int_{1-\alpha\delta}^{1} p(t, y)dy \leq C \times \alpha\delta\,,$$

which is not right if $C\alpha < 1$.



So, we impose

$$\alpha < \frac{1}{C}\,,$$

and then $\Delta e(t) = 0$.

$\frac{1}{2}$—Hölder property

In small time, W is reponsable for the pushing above the treshold, so

$$|e(t+h)-e(t)| \leq C_{1/2}h^{1/2}\,.$$

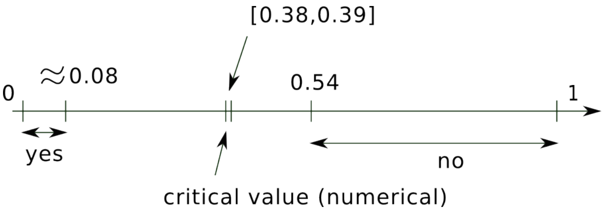
Iteration (up to T)

- Suppose $\mathbb{P}(X_0 \in dy) \leq \beta(1-y)dy$ ($y \in [1-\epsilon, 1[$), and suppose $p(0,.)$ is differentiable in 1, then we have a solution upto time $T_1(\beta, \epsilon)$ (those estimates depend on the a priori estimate $C_{1/2}$).
- We show that for a solution (X_t) on $[0;T]$, then $\mathbb{P}(X_t \in dy) \leq C_{\text{den}}(T)(1-y)dy$ (if y in some neighborhood of 0).
- We can then iterate the argument.

The unicity comes from the contraction property.

Conclusion

With $b = 0$ and $X_0 = 0.8$,



Particle system

N particles, start with $X_0^{i,N} \overset{\text{law}}{=} X_0$ $(\forall i)$ (i.i.d.)

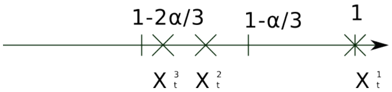
$$X_t^{i,N} = X_0^{i,N} + \int_0^t b(X_s^{i,N})ds + \frac{\alpha}{N} \sum_{j \neq i} M_t^{j,N} + W_t^i - M_t^{i,N}$$

with

- $M_t^{i,N} = \sum_{k \geq 0} \mathbb{1}_{[0,t]}(\tau_k^{i,N})$
- $\tau_0^{i,N} = 0$,
  $\tau^{i,N} = \inf\{t > \tau_{k-1}^{i,N} \ : \ X_{t-}^{i,N} + \frac{\alpha}{N} \sum_{j \neq i}(M_t^{j,N} - M_{t-}^{j,N}) \geq 1\}$

Non-unicity of the solution

There is a "logical loop" in the definition. Here is an example with $N = 3$.



"Physical" system/cascade of events

- $\Gamma_0 = \{i \in [N] \, : \, X^i_{t-} = 1\}$
- $\Gamma_1 = \{i \in [N] \backslash \Gamma_0 \, : \, X^i_{t-} + \alpha \frac{|\Gamma_0|}{N} \geq 1\}$
- $\Gamma_2 = \{i \in [N] \backslash (\Gamma_0 \cup \Gamma_1) \, : \, X^i_{t-} + \alpha \frac{|\Gamma_0 \cup \Gamma_1|}{N} \geq 1\}$
- and so on
- $\Gamma = \cup_{0 \leq k \leq N-1} \Gamma_k$

Set

$$X^i_t = \begin{cases} X^i_{t-} + \alpha \frac{|\Gamma|}{N} & \text{if } i \notin \Gamma \,, \\ X^i_{t-} + \alpha \frac{|\Gamma|}{N} - 1 & \text{if } i \in \Gamma \,. \end{cases}$$

Weak solution and synchronisation

Theorem

There exists a law μ such that, the canonical process z satisfies

1. *under μ, z_0 has the same law as X_0*
2. *under μ, $z_t - z_0 - \int_0^t b(z_s - m_s)ds - \alpha\langle\mu, m_t\rangle$ is a Brownian motion ($m_t = \lfloor (z_t)_+^* \rfloor$).*
3. *for all t, $\mu(\Delta m_t \leq 1) = 1$*
4. $\Delta\langle\mu, m\rangle_t = \inf\{\eta \geq 0 \,:\, \mu(x_{t-} + \alpha\eta \geq 1) \leq \eta\}$ $(= \sup\{\eta > 0 \,:\, \forall\eta' \leq \eta,\, \mu(x_{t-} + \alpha\eta' \geq 1) \geq \eta'\}\ )$

We then have solutions such that m_t can jump (under law μ), meaning there is a spike.

F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *Particle systems with a singular mean-field self-excitation. Application to neuronal networks*, Stochastic Process. Appl. **125** (2015), no. 6, 2451–2492. MR 3322871

François Delarue, James Inglis, Sylvain Rubenthaler, and Etienne Tanré, *Global solvability of a networked integrate-and-fire model of McKean–Vlasov type*, Ann. Appl. Probab. **25** (2015), no. 4, 2096–2133. MR 3349003