A general framework for stochastic traveling waves, with application to neural field equations.

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Joint work with J. Inglis

Outline

- Motivation: neural field equations
- Stochastic Differential Equations
- Tracking the Wavefront
- Short-time stability
- Long-time behaviour

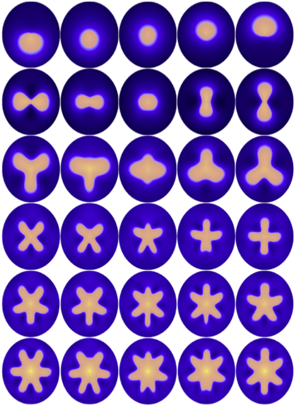
Motivation: neural field equations (I)

Spatio-temporal model for activity states in physical cortical slice

$$\partial_t u_t(x) = -u_t(x) + \int_{\mathbb{R}} w(x-y) F(u_t(y)) dy, \quad x \in \mathbb{R}.$$

- $u_t(x)$ is the activity of a typical neuron at position x in the cortical region modeled as $\mathbb{R}$
- $w \in L^1(\mathbb{R})$ represents the spatial distribution of neuronal synaptic connections
- F is the nonlinear gain function, typically taken to be a sigmoid

Neural field equations (II)



From Folias and Bressloff, Phys. Rev. Lett., 95, 2005

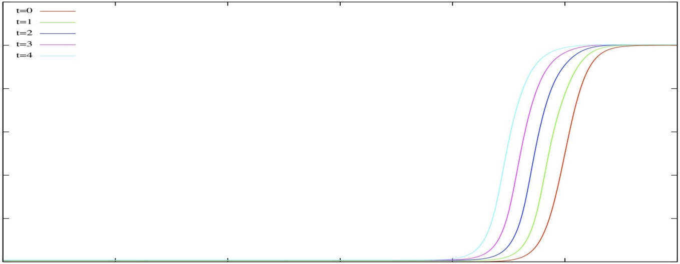
- Much studied in mathematical neuroscience literature
- Good models for describing signal propagation in cortical slices
- Propagation of activity in the brain important in characterizing neural disorders (e.g. epilepsy) as well as normal functionality
- Have been used to make a number of experimentally verified predictions

Motivation: neural field equations (III)

Main interest is the existence of **traveling front** solutions: $u_t(x) = \varphi_0(x - ct)$, $c \in \mathbb{R}$. Substituting into NF equation

$$0 = A\varphi_0 + f(\varphi_0), \qquad (*)$$

where $A\varphi_0 := c\varphi_0' - \varphi_0$ and $f(\varphi_0) := w * F(\varphi_0)$.



By translation invariance, $(\varphi_\alpha)_{\alpha \in \mathbb{R}}$ where $\varphi_\alpha = \varphi_0(\cdot + \alpha)$ is a family of solutions to (*).

Motivation: traveling pulses

A variation of the NF equation is

$$\begin{cases} \partial_t u_t = -u_t + w * F(u_t) - v_t \\ \partial v_t = \theta u_t - \epsilon v_t. \end{cases}$$

Can this time find a solution of the form $(\hat{u}(\cdot - ct), \hat{v}(\cdot - ct))$ such that $\varphi_0 = (\hat{u}, \hat{v}) \in [L^2(\mathbb{R})]^2$. By substitution, $\varphi_\alpha = \varphi_0(\cdot + \alpha)$ solves

$$0 = A\varphi_\alpha + f(\varphi_\alpha), \quad \alpha \in \mathbb{R},$$

where $A\varphi_\alpha = c\varphi'_\alpha + \begin{pmatrix} -1 & -1 \\ \theta & -\epsilon \end{pmatrix} \varphi_\alpha$, $f(\varphi_\alpha) = \begin{pmatrix} w * F(\hat{u}) \\ 0 \end{pmatrix}$.

Since $\hat{u} \in L^2(\mathbb{R})$, the solution is a **traveling pulse**, rather then a front.

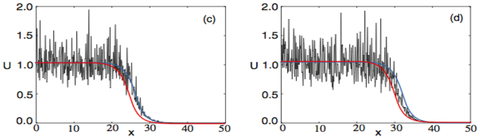
Motivation: stochastic neural field equations

Stochastic version on NF equation

$$du_t = [-u_t + w * F(u_t)]\, dt + dW_t^Q$$

where $(W_t^Q)_{t \geq 0}$ is $L^2(\mathbb{R})$-valued Q-Wiener process

- Studied in [Bressloff and Webber, 2012], [Kilpatrick and Ermentrout, 2013], and [Kruger and Stannat, 2014].
- Main idea: Decompose $u_t(x) = \varphi_0(x - ct - C(t)) + v_t(x)$ and try to choose $C(t)$ in a sensible way.



From Folias and Bressloff, SIAM J. Appl. Dyn. Syst., 11, 2012

General setting

E_0 be a Banach space, A and f linear and nonlinear operators respectively acting in E_0. Suppose that $\exists\ (\varphi_\alpha)_{\alpha\in\mathbb{R}} \subset E_0$ such that

$$0 = A\varphi_\alpha + f(\varphi_\alpha), \quad \alpha \in \mathbb{R}.$$

Let $H = [L^2(\mathbb{R}^d)]^N$ for $N, d \geq 1$, with standard i.p. $\langle \cdot, \cdot \rangle$. Let $E := \varphi_0 + H$, with topology inherited from H.

Assumptions on f and A:

(i) f is globally Lipschitz and $f'(u) \in L(H, H)$.

(ii) The operator A also acts in H and is the generator of a $\mathcal{C}_0$-semigroup on H.

General setting (2)

Assumptions on (φ_α):

(i) The derivatives (w.r.t norm in H) $[d^k/d\alpha^k]\varphi_\alpha \in H$ for $k \in \{1,2,3\}$. We will denote these derivatives by $\varphi'_\alpha, \varphi''_\alpha$, and φ'''_α respectively.

(ii) $\|\varphi'_\alpha\|, \|\varphi''_\alpha\|$, and $\|\varphi'''_\alpha\|$ are independent of α.

(iii) $\alpha \mapsto \varphi'_\alpha, \varphi''_\alpha$, and φ'''_α are all globally Lipschitz.

(iv) Some other technical assumptions satisfied by both waves and pulses.

General setting (3)

Consider the stochastic evolution equation

$$du_t = [Au_t + f(u_t)]dt + \varepsilon B(t)dW_t^Q, \qquad (1)$$

where $\varepsilon > 0$ and

Assumptions on the noise:

(i) $(W_t^Q)_{t\geq 0}$ is an H-valued Q-Wiener process with Q a bounded, symmetric, non-negative definite linear operator on H such that $\mathrm{Tr}(Q) < \infty$

(ii) $B : [0,\infty) \to L(H,H)$ is continuous, and $\|B(t)\|_{L(H,H)} \leq C \; \forall t \geq 0$.

(iii) $B(t)^* B(t) = \mathrm{Id}$ for all $t \geq 0$.

Proposition: existence and uniqueness

Suppose u_0 is such that $v_0^\alpha := u_0 - \varphi_0 \in H$. Then (1) has a unique solution $u_t = \varphi_0 + v_t^0$ where $(v_t^0)_{t\geq 0}$ is the unique weak (and mild) H-valued solution to

$$dv_t^0 = [Av_t^0 + f(\varphi_\alpha + v_t^0) - f(\varphi_0)]dt + \varepsilon B(t)dW_t^Q, \quad t \geq 0.$$

Tracking the wave

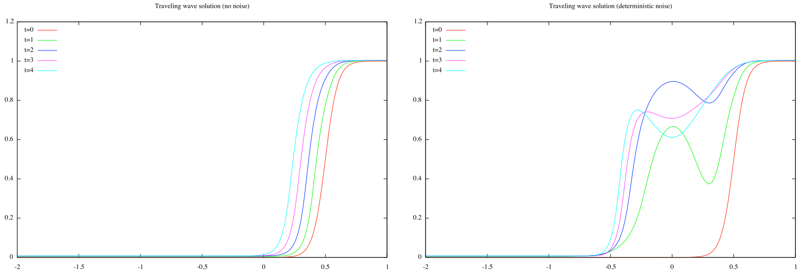
- $\exists\,!$ solution $(u_t)_{t\geq 0} \subset E$: $u_t = \varphi_0 + v_t^0$ where $(v_t^0)_{t\geq 0} \subset H$
- Question: How does the noise shift the wave front?
- Decompose solution as

$$u_t = \varphi_{\beta_t} + z_t, \quad t \geq 0.$$

$\rightarrow$ Would like to choose β_t to minimize $\alpha \mapsto \|u_t - \varphi_\alpha\|^2$.

- Can show that there always exists global minimizer, but may not be unique and may jump! Plus there may exist many local minima...

Tracking the front: illustration

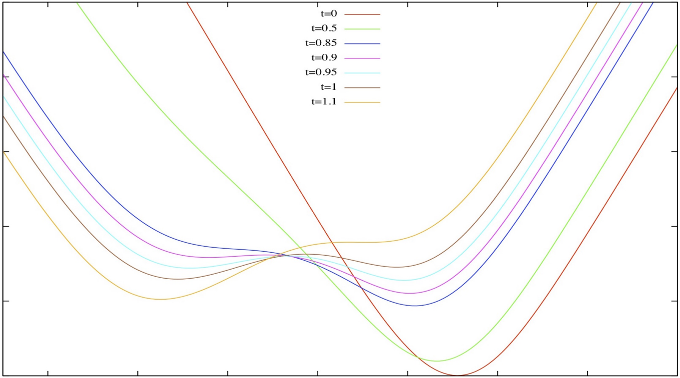


N.B. approximation is actually with 'deterministic noise' i.e. $(u_t)_{t\geq 0}$ solves

$$\partial_t u_t = -u_t + w * F(u_t)dt + \cos(t)e^{-10x^2}$$

Tracking the front: illustration

Numerical approximation of $\alpha \mapsto m(t,\alpha) := \|u_t - \varphi_\alpha\|^2$ in neural field case at different times:



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Local dynamics of the minimum (1)

We want an SDE for the evolution of a local minimum β_t, i.e.

$$\Delta(u_t, \beta_t) := \frac{1}{2}\frac{d}{d\beta_t}\left\|u_t - \varphi_{\beta_t}\right\|^2 = \langle u_t - \varphi_{\beta_t}, \varphi'_{\beta_t}\rangle = 0. \tag{2}$$

Note that $\langle u_t, \varphi'_{\beta_t}\rangle = \langle \varphi_0 + v_t^0, \varphi'_{\beta_t}\rangle = \langle \varphi_0, \varphi'_{\beta_t}\rangle + \langle v_t^0, \varphi'_{\beta_t}\rangle$.

Guess that $d\beta_t = \mu(t, \beta_t, u_t)dt + g(t, \beta_t, u_t)dW_t^Q$ (with $g \in L(H, H)$).
Applying Ito's Lemma to (2) we see that (formally)

$$\begin{aligned} d\Delta(u_t, \beta_t) &= \frac{\partial \Delta}{\partial u_t} B(t) dW_t^Q + \frac{\partial \Delta}{\partial \beta_t} g(t, \beta_t, u_t) dW_t^Q + \text{t.o.b.v} \\ &= \langle \varphi'_{\beta_t}, B(t)dW_t^Q\rangle + \langle u_t, \varphi''\rangle g(t, \beta_t, u_t) dW_t^Q + \text{t.o.b.v} \end{aligned}$$

Hence $g \cdot W = \frac{1}{\gamma(u_t, \beta_t)}\langle \varphi'_{\beta_t}, B(t)W\rangle$, where

$$\gamma(u_t, \beta_t) = -\langle u_t, \varphi''_{\beta_t}\rangle = \frac{1}{2}\frac{d^2}{d^2\beta_t}\left\|u_t - \varphi_{\beta_t}\right\|^2.$$

Local dynamics of the minimum (2)

SDE for dynamics of local minimum of $\alpha \mapsto m(u_t, \alpha) := \|u_t - \varphi_\alpha\|^2$:

$$d\beta_t = \mu(t, \beta_t, u_t)dt + \frac{1}{\gamma(u_t, \beta_t)}\langle \varphi_{\beta_t}, B(t)dW_t^Q \rangle, \quad t \geq 0.$$

Here

$$\mu(t, x, u_t) = \frac{1}{\gamma(u_t, \beta_t)}\langle u_t, A^* \varphi_x' \rangle - \langle \varphi_0, \varphi_0' \rangle + \frac{1}{\gamma^2(u_t, \beta_t)}\varepsilon^2 \langle B(t)QB^*(t)\varphi_x', \varphi_x'' \rangle + \frac{1}{\gamma^3(u_t, \beta_t)}\frac{\varepsilon^2}{2}\langle u_t - \varphi_x, \varphi_x''' \rangle \langle B(t)QB^*(t)\varphi_x', \varphi_x' \rangle.$$

Local Existence and Uniqueness for β_t

Thm: If $\frac{d}{d\beta_0} m(u_0, \beta_0) = 0$, then there exists a unique solution β_t to the previous SDE, for all times $t < \tau_\infty$. Here

$$\tau_\infty = \inf\{t > 0 : \gamma(\beta_t, u_t) = 0\}.$$

The solution satisfies

$$\frac{1}{2}\frac{d}{d\beta_t} m(u_t, \beta_t) = \langle u_t - \varphi_{\beta_t}, \varphi'_{\beta_t} \rangle = 0.$$

Under some technical assumptions, $\mathbb{P}(\lim_{t\to\tau_\infty} \beta_t \text{ exists}) = 1$.

Comparison to Previous Work

[Bressloff and Webber, 2012],[Kilpatrick and Ermentrout, 2013] Write $u_t = \varphi_0 + v_t^0$. Formally $|\beta_t|, \left\| v_t^0 \right\| \simeq O(\varepsilon)$, $\beta_0 = 0$.

$$\gamma(\beta_s, u_s) = \langle \varphi_0', \varphi_{\beta_t}' \rangle - \langle v_t^0, \varphi_{\beta_t}'' \rangle \simeq \left\| \varphi_0' \right\|^2 .$$

$$\beta_t \simeq \int_0^t \frac{\langle \mathcal{L}_0 v_s^0, \varphi_0' \rangle}{\left\| \varphi_0' \right\|^2} ds + \varepsilon \int_0^t \frac{\langle \varphi_0', B(s) dW_s^Q \rangle}{\left\| \varphi_0' \right\|^2} + \mathcal{O}(\varepsilon^2).$$

Here $\mathcal{L}_0 = A + f'(\varphi_0)$. Their equation is similar, with φ_0' replaced with Ψ_0, where $\mathcal{L}_0^* \Psi_0 = 0$.

[Kruger and Stannat,2013] Recall that $\Delta(u_t, \beta_t) = \frac{1}{2} \frac{d \left\| u_t - \varphi_{\beta_t} \right\|^2}{d\beta_t}$. They define β_t to satisfy the ODE, for some $M > 0$

$$\frac{d\beta_t}{dt} = -M\Delta(u_t, \beta_t).$$

Local stability

Behavior of $\|z_t\|^2 = \|u_t - \varphi_{\beta_t}\|^2$ and τ_∞ for small ε?

Spectral gap assump: Suppose $\mathcal{L}_\alpha := A + f'(\varphi_\alpha)$ has a spectral gap $b > 0$ independent of α.

Thm: $\exists C$ such that for any $t \in [0, \tau_\infty)$

$$d\|z_t\|^2 \leq (-b\|z_t\|^2 + C\|z_t\|^3)dt + \varepsilon^2 \left[\mathrm{Tr}(Q) - \frac{\|Q^{\frac{1}{2}}\varphi'_{\beta_t}\|^2}{\gamma(\beta_t, u_t)} \right] dt + 2\varepsilon\langle z_t, B(t)dW_t^Q\rangle.$$

Cor: If $\|z_0\|^2 + \varepsilon^{\frac{1}{2}} + 2b^{-1}\varepsilon^2\mathrm{Tr}(Q) < K$, ($K > 0$, independent of ϵ) then

$$\|z_t\|^2 \leq e^{-bt}(\|z_0\|^2 + \sqrt{\varepsilon}) + \varepsilon^2 C(1 - e^{-bt}),$$

for all $t < \rho_\varepsilon$, where

$$\rho_\varepsilon := \inf\left\{ t > 0 : 2\int_0^t e^{br/2}\langle z_r, B(r)dW_r^Q\rangle \geq \varepsilon^{-\frac{1}{2}} \right\}.$$

Motivation to Study the Long-Time Behaviour beyond τ_∞

Behavior of $\|z_t\|^2 = \|u_t - \varphi_{\beta_t}\|^2$ for large t, fixed ε?

Consider traveling wave solution to the deterministic equation

$$du_t = [Au_t + f(u_t)]dt.$$

Under some technical assumptions, [Chen,97] showed that there $\exists \kappa > 0$ and $\alpha, \mathcal{K} \in \mathbb{R}$ (depending on u_0) such that

$$\|u_t - \varphi_\alpha\|_\infty \leq \mathcal{K} \exp\left(-\kappa t\right).$$

Compare this to the 1-dimensional Ornstein-Uhlenbeck Process:

$$dy_t = -\kappa y_t dt + dW_t.$$

Reminder: the stochastic neural field equation

$$du_t = \left[c\frac{d}{dx}u_t - u_t + w * F(u_t) \right] dt + B(t)dW_t^Q$$

This is in the moving frame!!

If the noise is stationary, then we need $B(t) \cdot v(x) := v(x + ct)$.

Long time behaviour (neural field model)

Let β_t^* be a global minimum of $\alpha \mapsto m(t,\alpha) := \|u_t - \varphi_\alpha\|^2$. Let $z_t^* = u_t - \varphi_{\beta_t^*}$.

Thm: (In Progress) Subject to some technical assumptions on the noise, at any time t, almost surely β_t^* is the unique global minimum. Furthermore for all $r, s \in \mathbb{R}$, $r < s$

$$\|z_s^*\|^2 - \|z_r^*\|^2 = \int_r^s \Bigg(-2\,\|z_t^*\|^2 + 2\langle f(\varphi_{\beta_t^*} + z_t^*) - f(\varphi_{\beta_t^*}), z_t^*\rangle \\ + \varepsilon^2 \left[\mathrm{Tr}(Q) - \frac{\|Q^{\frac{1}{2}}\varphi'_{\beta_t^*}\|^2}{\gamma(\beta_t^*, v_t^0)} \right] \Bigg) dt + 2\varepsilon \int_r^s \langle z_t^*, B(t)dW_t^Q \rangle.$$

We believe that a sufficient condition on the noise should be $\langle v, Q_t v\rangle > 0$, for all $v \in H$, where

$$Q_t := \int_0^t B(s)^* Q B(s) ds.$$

Proof Sketch

Define the set E_δ: For $\delta \in (0,1)$ define $E_\delta \subset E$ by $u \in E_\delta \Leftrightarrow$

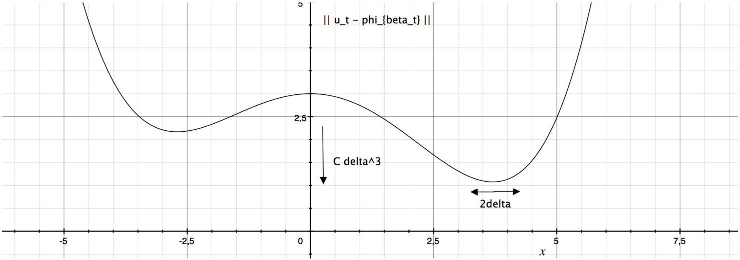
- $\|u\| \leq \delta^{-1}$
- $\exists$ a unique global minimum $\Gamma(u)$ of $\alpha \mapsto \|u - \varphi_\alpha\|^2$,
- for all $\alpha \in [\Gamma(u) - \delta, \Gamma(u) + \delta]$, $\gamma(\alpha, u) > \delta$,
- for all $\alpha \in \mathbb{R} \backslash [\Gamma(u) - \delta, \Gamma(u) + \delta]$, for some $C > 0$,

$$\|u - \varphi_\alpha\| > \left\|u - \phi_{\Gamma(u)}\right\| + C\delta^3.$$

For some $\delta \in (0,1)$, define the stopping times

$$\tau^{2k+1} = \inf\left\{t \geq \tau^{2k} : u_t \notin E_{\delta/2},\right\},$$
$$\tau^{2k+2} = \inf\left\{t \geq \tau^{2k+1} : u_t \in E_{\delta}.\right\}.$$

Minima of $\|u_t - \varphi_{\beta_t}\|$



Over the time intervals $[\tau^{2k}, \tau^{2k+1}]$, the SDE for β_t is well-defined (since γ has a positive lower bound). We find (using similar methods)

$$\left\|z^*_{\tau^{2k+1}}\right\|^2 - \left\|z^*_{\tau^{2k}}\right\|^2$$
$$= \int_{\tau^{2k}}^{\tau^{2k+1}} \Bigg(-2\left\|z^*_t\right\|^2 + 2\langle f(\varphi_{\beta^*_t} + z^*_t) - f(\varphi_{\beta^*_t}), z^*_t\rangle$$
$$+\varepsilon^2 \left[\mathrm{Tr}(Q) - \frac{\|Q^{\frac{1}{2}}\varphi'_{\beta^*_t}\|^2}{\gamma(\beta^*_t, v^0_t)} \right] \Bigg) dt + 2\varepsilon \int_{\tau^{2k}}^{\tau^{2k+1}} \langle z^*_t, B(t) dW^Q_t \rangle.$$

The basic idea is to find conditions such that $\tau^{2k+2} - \tau^{2k+1} \to 0$ almost surely.

Summary

- Proved that the local minimum of $\beta_t \to \|u_t - \varphi_{\beta_t}\|^2$ obeys an SDE over domains where the curvature $\gamma(u_t, \beta_t)$ is nonzero.
- Proved that the stopping time $\tau_\infty \to \infty$ as $\epsilon \to 0$
- Obtained a bound of the form, before a stopping time ρ_ϵ which $\to \infty$ as $\epsilon \to 0$.

$$\|z_t\|^2 \le e^{-bt}(\|z_0\|^2 + \sqrt{\varepsilon}) + \varepsilon^2 C(1 - e^{-bt}),$$

- (In progress) For any two times $r, s, > 0$, $r < s$, we have the inequality

$$\|z_s^*\|^2 - \|z_r^*\|^2 = \int_r^s \Bigg(-2\|z_t^*\|^2 + 2\langle f(\varphi_{\beta_t^*} + z_t^*) - f(\varphi_{\beta_t^*}), z_t^*\rangle$$
$$+\varepsilon^2 \left[\mathrm{Tr}(Q) - \frac{\|Q^{\frac{1}{2}}\varphi'_{\beta_t^*}\|^2}{\gamma(\beta_t^*, v_t^0)} \right] \Bigg) dt + 2\varepsilon \int_r^s \langle z_t^*, B(t) dW_t^Q \rangle.$$

Further Work

- Use a weighted inner product $\langle \cdot, \cdot \rangle_\alpha$ that shifts with φ_α.
- More investigation of the spectra / internal dynamics.
- Extend to systems with multiple degrees of freedom (such as spiral waves).
- Extend to systems where φ_α is periodic in α. (e.g. Zebra Stripes)

Acknowledgments

This is joint work with James Inglis (INRIA Sophia-Antipolis, France).

We thank P. Bressloff, W. Stannat, E. Lang and J. Kruger for interesting and helpful discussions.

We recommend that you look at the posters of E. Lang and J. Kruger!!