Uniform Propagation of Chaos in Mean Fields

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Let $(W^j)_{j\in\mathbb{Z}^+}$ be independent Wiener Processes, and $b_0, b_1, b_2 : \mathbb{R} \to \mathbb{R}$ measurable functions. Consider the stochastic differential equation

$$dX_t^j = \left(b_0(X_t^j) + \frac{1}{N}\sum_{k=1}^{N} b_1(X_t^j, X_t^k) \right) dt + b_2(X_t^j)dW_t^j. \tag{1}$$

Define the limit equation

$$d\bar{X}_t = \left(b_0(\bar{X}_t) + \int_{\mathbb{R}} b_1(\bar{X}_t, y)d\bar{\mu}_t(y) \right) dt + b_2(\bar{X}_t)dW_t^1, \tag{2}$$

where $\bar{\mu}_t$ is the law of $\bar{X}_t$. Classical Propagation of Chaos results typically revolve around the proof of identities of the form

$$\mathbb{E}\left[\sup_{t\leq T}\left(X_t^1 - \bar{X}_t\right)^2\right] \leq \frac{C}{N}, \tag{3}$$

where C is a constant independent of j and N, but dependent on $T > 0$. From the above result one may deduce the convergence of $(X_s^j)_{s\in[0,T]}$ (considered as a measure on the space of continuous functions on $[0,T]$) to $(\bar{X}_s)_{s\in[0,T]}$. However the constant C of convergence often grows in time T. Our result is an analogue of (3), but with a rate of convergence which is uniform in time. For $x, y \in \mathbb{R}$, and functions g and f to be specified further below, let $h(x,y) = g(x)g(y)f(x-y)$. Our chief result is that, under certain technical assumptions on b_0, b_1 and b_2, we obtain the inequality,

$$E\left[h(X_t^1, \bar{X}_t)\right] \leq CN^{-a}, \tag{4}$$

for constants $a, C > 0$ which are independent of t. f is smoothly differentiable, satisfying $f(x) = 0$ if and only if $x = 0$, the minimum of f on any closed set not containing the origin is strictly greater than zero, and $g \geq 1$. This work is essentially a generalization of [3], and various concentration inequality results such as [1]. In these works typically $g = 1$ everywhere, $a = 1$ and $f(z) = z^2$. Our result (4) allows the uniform propagation of chaos result to be applied in more biologically realistic contexts. Neuroscientific models are only experimentally validated over a finite parameter range. Accordingly we expect that the probability of X_t^j assuming extremely large or extremely small values to be very low. g is taken to be identically one in some compact domain $\mathcal{D}$; it is used to modulate the behavior for when $|X_t^1|$ or $|\bar{X}_t|$ are asymptotically large; events which we expect to be very improbable. f modulates the rate of convergence for when $X_t^1 - \bar{X}_t \in \mathcal{D}$.

Indeed the uniform propagation of chaos result is essentially due to the stabilizing effect of the internal dynamics (b_0 term) outweighing the destabilizing effect of the inputs from other neurons (b_1 term) and the noise (b_2 term). b_0 is monotonically decreasing. In [3], it was assumed that the gradient of b_0 is at least linear. However it is not at all clear that the decay resulting from the internal dynamics term is always this strong for large values of X_t^j; the models have not really been adequately tested in this range. Our more abstract setup does not require the decay to be linear: indeed the decay could be sub linear or even bounded; all that is required is that in the asymptotic limit the decay from b_0 dominates the destabilizing effects of b_1 and b_2. Another improvement of our model over [3] is that we consider multiplicative noise (i.e. $b_2 \neq 1$). This is much more biologically realistic because we expect the noise term $\int_0^t b_2(X_s^j)dW_s^j$ to be of decreasing influence as $|X_t^j|$ gets large. This is because we do not expect the noise to be so great that it pushes X_t^j far beyond the experimental range.

We apply these results to some neuronal models, illustrating the convergence.

References

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