From smooth firing rate functions to the Heaviside function in homogenized neural field models: The case of bump solutions

Evgenii Burlakov, Norwegian University of Life Sciences, email: evgenii.burlakov@nmbu.no
Arcady Ponosov, Norwegian University of Life Sciences, email: arkadi.ponossov@nmbu.no
John Wyller, Norwegian University of Life Sciences, email: john.wyller@nmbu.no

Cortical networks are often investigated by using firing rate neural field models. The most well-known and simplest model describing the neural field dynamics is the Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int\limits_{\Omega} \omega(x-y) f(u(t,y))dy, \quad t \geq 0, x \in \Omega \subseteq R^n. \tag{1}$$

Here $u(t,x)$ denotes the activity of a neural element at time t and position x. The connectivity function $\omega(x)$ determines the coupling between the elements and the non-negative function $f(u)$ gives the firing rate of a neuron with activity u. Typically f is a smooth function that has sigmoidal shape. Particular attention is usually given to the localized stationary, i.e., time-independent, solutions to (1) (so-called "bumps"), as they correspond to normal brain functioning. Faugeras et al [2] proved existence and uniqueness of the stationary solution to (1) as well as obtained conditions for this solution to be absolutely stable, for the case of a bounded Ω. A common simplification of (1) consists of replacing the smooth firing rate function by the Heaviside function. This replacement simplifies numerical investigations of the model as well as allows to obtain closed form expressions for some important types of solutions (see e.g. [1, 3]). Stability of the stationary solutions to (1) is usually assessed by the Evans function approach (see e.g. [4]). However, no rigorous mathematical justification of the passage from a smooth to discontinuous firing rate functions in the framework of neural field models was given until the work by Oleynik et al [5], where continuous dependence of the 1-bump stationary solution to (1) under the transition from a smooth firing rate function to the Heaviside function was proved in the 1-D case. The pioneering work by Coombes et al [6] introduced the following development of (1): the homogenized Amari model describing the neural field dynamics on both macro- and micro-levels

$$\partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int\limits_{\Omega}\int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f(u(t,y_c,y_{\mathrm{f}}))dydy_{\mathrm{f}}, \quad t \geq 0, \ x \in \Omega, \ x_{\mathrm{f}} \in \mathcal{Y} \subset R^k. \tag{2}$$

Here x_f is the fine-scale spatial variable and $\mathcal{Y}$ is some torus. Existence and stability of the stationary bump solutions to (2) in 1-D was investigated in [7] for the case of the Heaviside firing rate function. In the present research we extend the results of [5] to the homogenized Amari model and, in addition to the single bumps in 1-D, consider symmetric 2-bumps in 1-D and rotationally symmetric bumps in 2-D. We formulate the following two theorems: the theorem on continuous dependence of the solutions to (2) under the transition from a smooth firing rate function to the Heaviside function and the theorem on solvability of the equation (2) based on topological degree theory. The topological degree has been calculated for the single bump solutions in 1-D, symmetric 2-bump solutions in 1-D and rotationally symmetric bumps in 2-D.

References

[1] S. Amari, Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields, *Biol. Cybernetics* (27) pp. 77-87, 1977.

[2] O. Faugeras, R. Veltz, F. Grimbert, Persistent Neural States: Stationary Localized Activity Patterns in Nonlinear Continuous n-Population, q-Dimensional Neural Networks, *Neural Computation* 21 (1), pp. 147-187, 2009.

[3] C.R. Laing, W.C. Troy, Two-bump solutions of Amari-type models of neuronal pattern formation, *Physica D* (178) pp. 190-218, 2003.

[4] S. Coombes, M.R.Owen, Evans functios for integral field equations with Heavisite firing rate function, *SIAM Journal on Applied Dynamical Systems* (34) pp. 574-600, 2004.

[5] A. Oleynik, A. Ponosov, J. Wyller, On the properties of nonlinear nonlocal operators arising in neural field models, *Journal of Mathematical Analysis and Applications* (398) pp. 335-351, 2013.

[6] S. Coombes, C. Laing, H. Schmidt, N. Svanstedt, J. Wyller, Waves in random neural media, *Discrete and Continuous Dynamical Systems, Series A* (32) pp 2951-2970, 2011.

[7] N. Svanstedt, J. Wyller, E. Malyutina, A one population Amari model with periodic microstructure, *Nonlinearity* (27) pp. 1391–1417, 2014.