Analysis and Approximation of Stochastic Nerve Axon Equations

Martin Sauer, TU Berlin, sauer@math.tu-berlin.de
Wilhelm Stannat, TU Berlin, stannat@math.tu-berlin.de

Stochastic partial differential equations arise as a model in many scientific fields, in particular in mathematical neuroscience they may be used to describe neurons with a spatial extent. However, in most applications it is the case that the examples, despite a very simple structure of its nonlinearities, fail to fit the assumptions (e. g. global Lipschitz condition on nonlinearities) of the majority of mathematical publications concerning the well-posedness and numerical approximation of such equations.

In this talk, we consider spatially extended, conductance based neuronal models with noise. These are described by a stochastic reaction diffusion equation with additive noise coupled to a control variable with multiplicative noise but no diffusion. The spatial geometry is a simple one dimensional domain $(0, L)$ approximating the axon. This is most accurate in the case of a long axon, shaped as a cylinder with constant diameter. Thus, the equations are realized as stochastic evolution equations on the space $H \times H^d$, $H = L^2(0, L)$, $H^d = \otimes_{i=1,\dots,d} H$.

$$\begin{aligned} \mathrm{d}U(t) &= \Big(AU(t) + f\big(U(t), \mathbf{X}(t)\big)\,\mathrm{d}t + B\,\mathrm{d}W(t), \\ \mathrm{d}X_i(t) &= f_i\big(U(t), X_i(t)\big)\,\mathrm{d}t + B_i\big(U(t), \mathbf{X}(t)\big)\,\mathrm{d}W_i(t), \quad 1 \le i \le d. \end{aligned} \tag{1}$$

Here, $\mathbf{x} = (x_i)_{1 \le i \le d}$, A is the Neumann laplacian on H and W, W_i is a family of independent cylindrical Wiener processes on H. Depending on the properties of f and f_i we depict two cases:

monotonocity

The nonlinear part of the drift of equation (1) satisfies a one-sided Lipschitz condition, hence the drift as a whole can be seen as a monotone operator. Under this assumption we prove well-posedness in the variational framework for stochastic evolution equations in [1].

Moreover, we consider strong convergence of the finite differences approximation in space and derive convergence with an implicit rate depending on the regularity of the exact solution. This can be made explicit if the variational solution has more than its canonical spatial regularity. As an application, spatially extended FitzHugh-Nagumo systems with noise are considered.

"conditioned" monotonocity

The famous Hodgkin-Huxley model for the propagation of an action potential along the neuron's axon, which is essentially the basis for all subsequent conductance based models for active nerve cells, unfortunately does not fit into the above scenario. In particular, the nonlinear drift in each equation is only monotone if the other variables are fixed, thus conditioned on the fixed path $\mathbf{X}$ the equation for U is well-posed and vice versa. In [2], we present a new method to derive crucial pathwise L^∞-bounds for the solutions U and $\mathbf{X}$, respectively, that do not rely on embedding theorems. These play an essential role in constructing solutions to the entire system via a fixed point iteration, that allows to prove existence and uniquness of variational solutions.

Again, we consider a spatial discretization via finite differences and study the strong approximation error. With the help of the pathwise L^∞-bounds, we derive explicit error estimates, in particular a pathwise convergence rate of $\sqrt{1/n}-$ and a strong convergence rate of $1/n$ in special cases. As applications, the Hodgkin-Huxley and FitzHugh-Nagumo systems with noise are considered.

References

[1] M. Sauer and W. Stannat. Lattice Approximation for Stochastic Reaction Diffusion Equations with One-Sided Lipschitz Condition. *Math. Comp.*, 84:743-766, 2015.

[2] M. Sauer and W. Stannat. Analysis and Approximation of Stochastic Nerve Axon Equations. *Arxiv preprint*, arXiv:1402.4701, 2014.