Rigorous results on robust traveling waves in periodically-forced chains of simple type-I oscillators

Stanislav M. Mintchev, The Cooper Union, USA, mintchev@cooper.edu
Bastien Fernandez, CNRS - Aix-Marseille Université, FR, fernandez@cpt.univ-mrs.fr

Emergent behavior and pattern formation continue to represent a broad area of scientific inquiry. Within this vast field, the generation and persistence of traveling wave solutions (TWs) in neural ensembles is a well-studied topic of mathematical neuroscience, where it finds applications to signal transmission and propagation of electrical impulses in brain tissues. Nevertheless, mathematically-rigorous studies of existence and stability for TWs in spatially extended systems have been limited primarily to models that feature *monotonicity* of the profile dynamics (this property is analogous to the maximum principle for parabolic PDE, and is typically present when ensembles are coupled diffusively); for such models, there have been extensive results on existence and asymptotic stability of waves (cf. [4, 6]).

In the present study [1], we are concerned with the emergence of stable TWs in *non-monotone* unidirectional chains of phase oscillators forced at their boundary node. The interaction in this model is motivated by the pulse-response paradigm for type-I phase oscillators receiving brief stimuli that are modulated by a phase response curve (PRC) prior to being incorporated into the frequency dynamics [2]. Numerical results in previous studies have indicated that such systems exhibit stable families of TWs and can be tuned to such a regime with a plain uniform-frequency forcing signal [3, 5]. With the goal of understanding this behavior, we propose a model that retains the qualitative features of type-I pulse-response interaction while yielding the added benefit that it is amenable to mathematical analysis. The oscillators of this reduced model take turns occupying "firing" and "quiescent" regimes; all the while, they alternate between having the ability to "hear" each other (i.e., receive signals from other members of the network) and being "deaf" to external stimulation. In brief, the dynamics is given by the following coupled ODEs

$$\frac{d\theta_s}{dt} = \omega + \epsilon\Delta(\theta_s)\delta(\theta_{s-1}),\ \forall s = 1, 2, \cdots \tag{1}$$

where $\theta_s \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$; the stimulus δ and the PRC Δ are piecewise-constant on $\mathbb{T}$, taking either the value 1 ("on") or 0 ("off"). The model's three natural parameters are the coupling strength ϵ and the sizes of the supports of δ and Δ within $\mathbb{T}$. Without loss of generality, one takes $\omega = 1$ by rescaling the time variable t. The function $t \mapsto \theta_0(t)$ is given and plays the role of the forcing signal (external stimulus) at site $s = 0$; it is assumed to be continuous, increasing and periodic (*viz.* there exists $\tau \in \mathbb{R}^+$ such that $\theta_0(t+\tau) = \theta_0(t) + 1$ for all $t \in \mathbb{R}^+$).

In this setting, we prove the existence of stable TWs for all parameter regimes of the system. We establish a codimension-1 global Lyapunov stability result for these waves, as well as a result in the direction of explaining the tuning phenomenology described above: waves are proved to emerge even when the system is forced using an a priori unadapted signal, for instance, a uniform forcing. We draw special attention to the point that monotonicity is violated for this model, as well as the motivating systems of pulse-response interactions for type-I oscillators. From this perspective, the study provides new insights into the mechanisms for nonlinear wave phenomena that occur in models motivated by neuroscience.

References

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