Asymptotic analysis of stochastic travelling waves in stochastic neural field equations

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Neural field equations are typically used to model the spatio-temporal evolution of a population of neurons, and arise when one takes a spatial limit of a discrete spatial network. The resulting nonlinear integro-differential equation describes the evolution in time of the voltage $u(t,x)$ across a neuron at time t located at position $x \in \Omega$, where Ω is some subset of Euclidean space representing the cortical region being modeled. More specifically, the classical neural field equation has the following form:

$$\partial_t u(t,x) = -u(t,x) + \int_\Omega w(x-y)S(u(t,y))dy, \qquad t \geq 0, x \in \Omega,$$

subject to initial and boundary conditions. Here $w : \Omega \to \mathbb{R}$ is a connectivity function representing how the activity in one region of Ω affects the activity in another region (also called the neural field kernel), and $S : \mathbb{R} \to \mathbb{R}$ is a nonlinear bounded function, typically taken to be a sigmoid.

One reason these equations have been the subject of much interest over the past decade is that (under some conditions and when $\Omega = \mathbb{R}$) they give rise to travelling wave solutions, that is solutions of the form $\hat{u}(x-ct)$ for some constant speed $c \in \mathbb{R}$ which are such that $\lim_{\xi \to \pm\infty} \hat{u}(\xi) = \hat{u}_\pm$ with $\hat{u}_+ < \hat{u}_-$ (see [1] for a review).

More recently several authors have become interested in stochastic versions of these equations, which are expressed in the following form

$$du_t = [-u_t + w * S(u_t)]\, dt + \varepsilon dW_t, \quad t \geq 0,$$

where as usual $*$ denotes convolution, $\varepsilon > 0$ and $(W_t)_{t\geq 0}$ is an $L^2(\mathbb{R})$-valued Wiener process. The existence and uniqueness of a solution $(u_t)_{t\geq 0}$ (which takes its values in a space of functions on $\mathbb{R}$) to such an equation can be made rigorous ([3, 4]). The ansatz in [2] was then that the solution can be decomposed as

$$u_t(x) = \hat{u}(x - ct - C(t)) + v_t(x), \quad t \geq 0, x \in \mathbb{R}, \tag{1}$$

where the process $C(t)$ represents the amount that the noise displaces the fixed wave profile $\hat{u}$ from its uniformly translating mean position. This decomposition was moreover made rigorous in [4], where the displacement $C(t)$ was chosen so as to minimize the $L^2(\mathbb{R})$-distance between the solution u_t and all possible translations of $\hat{u}$. However, this did not account explicitly for the conclusion in [2] that the displacement $C(t)$ should be of Brownian nature.

In this work we therefore take an alternative approach to try and rigorously justify the choice of $C(t)$ made in [2]. The key idea is that we choose $(C(t))_{t\geq 0}$ to be the stochastic process such that the correction process $(v_t)_{t\geq 0}$ in (1) has no noisy component in the direction of the traveling wave, thus ensuring that all displacement in this direction is captured by $C(t)$. With this choice of $(C(t))_{t\geq 0}$ we identify two regimes depending on the nature of the noise: one where the wave decays to a shifted travelling wave solution, and another where the travelling wave profile is lost if the noise is too large. In the process we develop the existing deterministic travelling wave theory ([5]) in a stochastic setting, leading to some general results.

References

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